Solution to Homework Assignment No. 4

1. (a) $\begin{cases} C = 0 \\ C = 8 \\ C = 8 \\ C = 20 \end{cases} \Longrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [C] = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \Longrightarrow \boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \widehat{\boldsymbol{x}} = [C], \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.$

$$\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [C] = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \Longrightarrow C = 9.$$

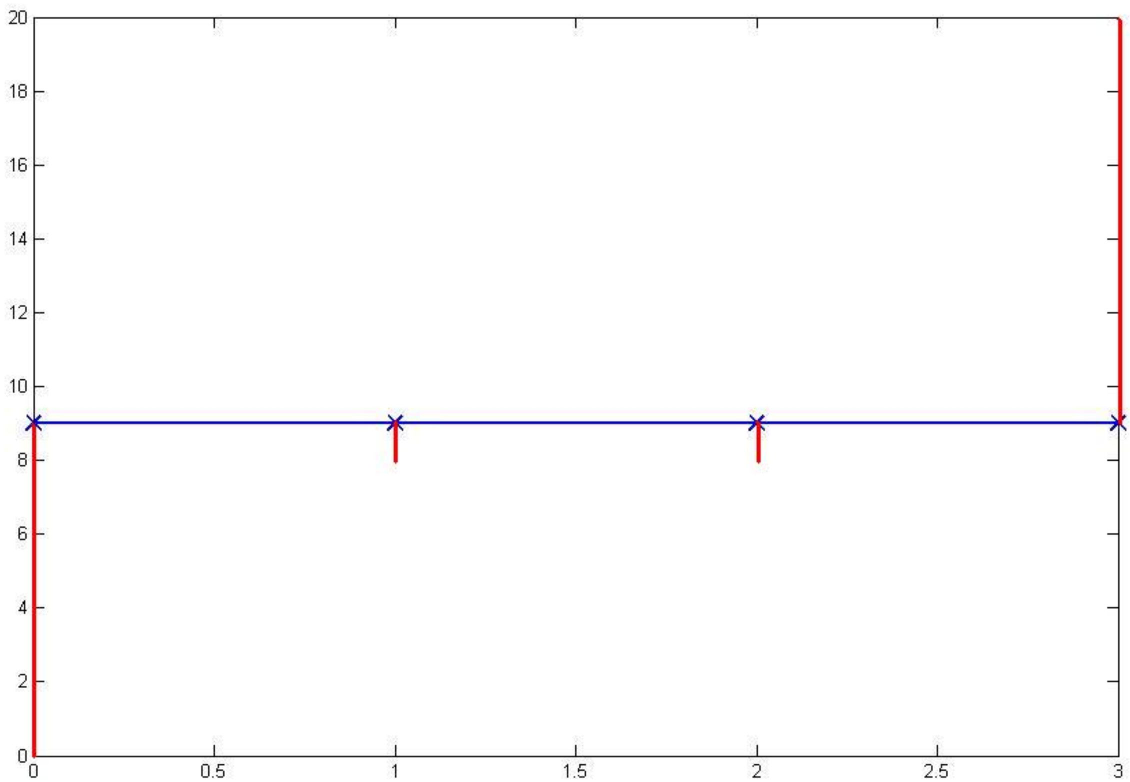

Figure 1: The blue line is the horizontal line and the red segments are the errors.

(b) Let

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.$$

Applying the formula for projection onto a line gives

$$\widehat{x} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\8\\8\\20\end{bmatrix}}{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} = \frac{36}{4} = 9.$$

We can obtain

$$\boldsymbol{p} = \widehat{x}\,\boldsymbol{a} = \begin{bmatrix}9\\9\\9\\9\end{bmatrix}, \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix}-9\\-1\\-1\\11\end{bmatrix}.$$

Note that $\boldsymbol{p}$ and $\boldsymbol{e}$ are orthogonal since

$$\boldsymbol{p}^T\boldsymbol{e} = \begin{bmatrix}9 & 9 & 9 & 9\end{bmatrix}\begin{bmatrix}-9\\-1\\-1\\11\end{bmatrix} = -81 - 9 - 9 + 99 = 0.$$

Also

$$\|\boldsymbol{e}\| = \sqrt{(-9)^2 + (-1)^2 + (-1)^2 + 11^2} = 2\sqrt{51}.$$

The result here is the same as that in (a).

2. $$\begin{cases} C + D(-2) = 4 \\ C + D(-1) = 2 \\ C + D(0) = -1 \\ C + D(1) = 0 \\ C + D(2) = 0 \end{cases} \Longrightarrow \boldsymbol{A} = \begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}, \quad \widehat{\boldsymbol{x}} = \begin{bmatrix}C\\D\end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix}.$$

We can have

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\-2 & -1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 0\\0 & 10\end{bmatrix}$$

and

$$\boldsymbol{A}^T\boldsymbol{b} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\-2 & -1 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}5 & -10\end{bmatrix}.$$

Therefore,

$$\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b} \Longrightarrow \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 & -10 \end{bmatrix} \Longrightarrow C = 1, \quad D = -1.$$

And the best line is $1 - t$.

3. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ denote the independent columns and $\boldsymbol{q_1}$, $\boldsymbol{q_2}$, $\boldsymbol{q_3}$ denote the desired orthonormal ones. By the Gram-Schmidt process, we can have

$$\begin{aligned} \boldsymbol{q_1} &= \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \\ \boldsymbol{q_2} &= \frac{\boldsymbol{b} - (\boldsymbol{q_1}^T\boldsymbol{b})\boldsymbol{q_1}}{\|\boldsymbol{b} - (\boldsymbol{q_1}^T\boldsymbol{b})\boldsymbol{q_1}\|} \\ \boldsymbol{q_3} &= \frac{\boldsymbol{c} - (\boldsymbol{q_1}^T\boldsymbol{c})\boldsymbol{q_1} - (\boldsymbol{q_2}^T\boldsymbol{c})\boldsymbol{q_2}}{\|\boldsymbol{c} - (\boldsymbol{q_1}^T\boldsymbol{c})\boldsymbol{q_1} - (\boldsymbol{q_2}^T\boldsymbol{c})\boldsymbol{q_2}\|}. \end{aligned}$$

Now $\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Substitute $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}$ and we can get $\boldsymbol{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{q_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{q_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{q_1} & \boldsymbol{q_2} & \boldsymbol{q_3} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_1}^T\boldsymbol{a} & \boldsymbol{q_1}^T\boldsymbol{b} & \boldsymbol{q_1}^T\boldsymbol{c} \\ 0 & \boldsymbol{q_2}^T\boldsymbol{b} & \boldsymbol{q_2}^T\boldsymbol{c} \\ 0 & 0 & \boldsymbol{q_3}^T\boldsymbol{c} \end{bmatrix}$$

which gives

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

4. It can be checked that the columns of $\boldsymbol{Q}$ are orthogonal if $c \neq 0$. We only have to find c for the lengths of each column vector to be one. We can have $c^2 + (-c)^2 + (-c)^2 + (-c)^2 = 1$, which gives $c = \pm 1/2$. Here we choose $c = 1/2$.

Now the first column $\boldsymbol{q_1} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$, and the second column $\boldsymbol{q_2} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$. The projection of $\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ onto the first column gives

$$\boldsymbol{p} = (\boldsymbol{q_1}^T\boldsymbol{b})\boldsymbol{q_1} = -\boldsymbol{q_1} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The projection of $\boldsymbol{b} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$ onto the first two columns yields

$$\begin{aligned}\boldsymbol{p}' &= (\boldsymbol{q_1}^T\boldsymbol{b})\boldsymbol{q_1} + (\boldsymbol{q_2}^T\boldsymbol{b})\boldsymbol{q_2} = -\boldsymbol{q_1} - \boldsymbol{q_2} \\ &= \frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.\end{aligned}$$

5. We can perform some row operations on $\boldsymbol{A}$ without changing the determinant:

$$\boldsymbol{A} = \begin{bmatrix}1 & -4 & 5\\2 & -8 & 10\\3 & -12 & 15\end{bmatrix} \implies \boldsymbol{A}' = \begin{bmatrix}1 & -4 & 5\\0 & 0 & 0\\3 & -12 & 15\end{bmatrix} \text{ (row 2 subtracts } 2 \cdot \text{row 1)}$$

We therefore have

$$\det \boldsymbol{A} = \det \boldsymbol{A}' \text{ (by Rule 5)} = 0 \text{ (by Rule 6)}.$$

As for $\det \boldsymbol{K}$, we first observe that $\boldsymbol{K}^T = -\boldsymbol{K}$. Therefore,

$$\det \boldsymbol{K} = \det \boldsymbol{K}^T \text{ (by Rule 10)} = \det(-\boldsymbol{K}) = (-1)^3 \det \boldsymbol{K} \text{ (by Rule 3)}$$

which gives

$$\det \boldsymbol{K} = 0.$$

6. Utilizing the rules of determinants, we can have

$$\begin{aligned}&\det \boldsymbol{L} = 1 \cdot 1 \cdot 1 = 1\\ &\det \boldsymbol{U} = 3 \cdot 2 \cdot (-1) = -6\\ &\det \boldsymbol{A} = \det \boldsymbol{L} \cdot \det \boldsymbol{U} = 1 \cdot (-6) = -6.\end{aligned}$$

Since

$$\boldsymbol{L}^{-1}\boldsymbol{L} = \boldsymbol{I} \text{ and } \boldsymbol{U}^{-1}\boldsymbol{U} = \boldsymbol{I}$$

we can have

$$\det \boldsymbol{L}^{-1} = 1/\det \boldsymbol{L} = 1$$

and

$$\det \boldsymbol{U}^{-1} = 1/\det \boldsymbol{U} = -1/6.$$

Therefore,

$$\det(\boldsymbol{U}^{-1}\boldsymbol{L}^{-1}) = \det \boldsymbol{U}^{-1} \cdot \det \boldsymbol{L}^{-1} = -1/6.$$

Also

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

which gives

$$\boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{A} = \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{L}\boldsymbol{U} = \boldsymbol{I}.$$

Hence,

$$\det(\boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{A}) = \det \boldsymbol{I} = 1.$$

7. (a) Let $E_n = |\boldsymbol{A_n}|$. Thus $\boldsymbol{A_n}$ is an n by n matrix. First observe that, for $n \geq 3$,

$$E_n = \begin{vmatrix} 1 & 1 \quad 0 \quad \cdots \quad 0 \\ 1 & \\ 0 & \boldsymbol{A_{n-1}} \\ \vdots & \\ 0 & \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \quad 0 \cdots \quad 0 \\ 1 & 1 & 1 \quad 0 \cdots \quad 0 \\ 0 & 1 & \\ 0 & 0 & \boldsymbol{A_{n-2}} \\ \vdots & \vdots & \\ 0 & 0 & \end{vmatrix}.$$

Applying the cofactor formula for the first row, we can have

$$\begin{aligned} E_n &= 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-1}}| + 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 \quad 0 \quad \cdots \quad 0 \\ 0 & \\ 0 & \boldsymbol{A_{n-2}} \\ \vdots & \\ 0 & \end{vmatrix} \\ &= E_{n-1} - 1 \cdot (-1)^{1+1} |\boldsymbol{A_{n-2}}| \text{ (apply the cofactor formula for the first column)} \\ &= E_{n-1} - E_{n-2}. \end{aligned}$$

(b) We recursively use the result in (a) and can obtain

$$\begin{aligned} E_3 &= E_2 - E_1 = 0 - 1 = -1 \\ E_4 &= E_3 - E_2 = -1 - 0 = -1 \\ E_5 &= E_4 - E_3 = -1 - (-1) = 0 \\ E_6 &= E_5 - E_4 = 0 - (-1) = 1 \\ E_7 &= E_6 - E_5 = 1 - 0 = 1 \\ E_8 &= E_7 - E_6 = 1 - 1 = 0. \end{aligned}$$

Also

$$E_9 = E_8 - E_7 = 0 - 1 = -1 = E_3 \text{ (repeated)}.$$

(c) Since from (b) we can find that the sequence $\{E_n\}$ is periodic of period 6, we have $E_{100} = E_4 = -1$.

8. We first write

$$\det \boldsymbol{A} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix}.$$

Applying the big formula, we can split det $\boldsymbol{A}$ into:

$$\begin{aligned}
\det \boldsymbol{A} &= a_{11}a_{22}a_{33}a_{44}\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} + a_{11}a_{22}a_{43}a_{34}\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{vmatrix} + a_{11}a_{32}a_{23}a_{44}\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{vmatrix} \\
&\quad + a_{21}a_{12}a_{33}a_{44}\begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} + a_{21}a_{12}a_{43}a_{34}\begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{vmatrix} \\
&= 2\cdot 2\cdot 2\cdot 2\cdot 1 + 2\cdot 2\cdot(-1)\cdot(-1)\cdot(-1)^1 \\
&\quad + 2\cdot(-1)\cdot(-1)\cdot 2\cdot(-1)^1 + (-1)\cdot(-1)\cdot 2\cdot 2\cdot(-1)^1 \\
&\quad + (-1)\cdot(-1)\cdot(-1)\cdot(-1)\cdot(-1)^2
\end{aligned}$$

which is equal to

$$16 - 4 - 4 - 4 + 1.$$